Radial point interpolation based finite difference method for mechanics problems

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SUMMARY

A radial point interpolation based finite difference method (RFDM) is proposed in this paper. In this novel method, radial point interpolation using local irregular nodes is used together with the conventional finite difference procedure to achieve both the adaptivity to irregular domain and the stability in the solution that is often encountered in the collocation methods. A least-square technique is adopted, which leads to a system matrix with good properties such as symmetry and positive definiteness. Several numerical examples are presented to demonstrate the accuracy and stability of the RFDM for problems with complex shapes and regular and extremely irregular nodes. The results are examined in detail in comparison with other numerical approaches such as the radial point collocation method that uses local nodes, conventional finite difference and finite element methods. Copyright © 2006 John Wiley & Sons, Ltd.

KEY WORDS: generalized finite difference method; meshfree method; radial basis function; radial point interpolation; least-square; numerical analysis

1. INTRODUCTION

Meshfree methods that use irregular nodes in local supporting domain in the approximation of field variables are being actively developed as a powerful numerical tool for various engineering applications (see, e.g. References [1-3]). The existing meshfree methods may be classified into three major categories according to their formulation procedures of discretizing the governing equations [2]: the methods based on a variational principle or a weak form of system equations (short for meshfree weak-form method), the methods based on the strong form of governing

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equations (short for meshfree strong-form method), and methods based on both weak and strong forms (short for meshfree weak-strong-form method). Currently, the meshfree weak-form method is most widely used meshfree techniques due to its excellent stability. It includes the element free Galerkin (EFG) method [4], reproducing kernel particle (RKP) method [5], meshless local Petrov–Galerkin (MLPG) method [6], and point interpolation methods [7–11]. The use of global or local integrations to establish the discrete equations is a common feature of the meshfree weak-form methods. The integrations have significant effects on providing stability, accuracy and convergence. However, the formulation procedures are relatively more complicated and more difficult to be implemented due to the need for background integrations.

Meshfree strong-form methods are regarded as a truly meshfree method as mesh is required for neither field variable approximation nor integration. The formulation procedure of the strong-form methods is relatively simple and straightforward, compared with the weak-form methods. Smoothed particle hydrodynamics (SPH) [12, 13] and the generalized finite difference method (GFDM) [14–18] may be under this category. Radial point collocation method (RPCM) is also a meshfree strong-form method (see, e.g. References [19–21]) formulated using radial basis functions and nodes in local supporting domains. Like other strong-form methods, the RPCM suffers from problems of instability (e.g. Reference [2]). Poor accuracy and instability issues often arise, especially when Neumann boundary conditions exist. This is particularly true for solid mechanics problems with force boundary conditions. The system equations behave very much like ill-posed inversed problems (e.g. Reference [22]). Several techniques have been proposed to overcome these shortcomings in the meshfree strong-form methods. Examples include the finite point method [23], Hermite-type collocation method (e.g. References [19, 20]), fictitious point approach (e.g. Reference [21]), stabilized least-squares RPCM (LS-RPCM) [24] and meshfree weak-strong (MWS) form method [25, 26].

Based on the aforementioned existing work, this paper presents a radial point interpolation based finite difference method (RFDM) as an alternative meshfree strong-form method. In this novel method, the point interpolation using radial basis functions and nodes in local support domain is incorporated into classical finite difference method (FDM) for stable solutions to partial differential equations defined in a domain that is represented by a set of irregularly distributed nodes. A least-square technique is adopted to acquire a system matrix of good properties including symmetry and positive definiteness, which helps greatly in solving the resulting set of algebraic system equations more efficiently and accurately by standard solver such as the Cholesky solver. The proposed RFDM can effectively avoid the instability in conventional collocation methods, while retaining the feature of simplicity in formulation procedures with little additional computational cost.

This paper is organized as follows. In Section 2, the standard radial point collocation method is briefly introduced. Section 3 gives theoretical formulation of the RFDM. Several numerical examples are presented in Section 4, and conclusions are drawn in Section 5.

2. RADIAL POINT COLLOCATION METHOD

2.1. Radial point interpolation

Radial basis functions (RBFs) are useful for surface fitting based on arbitrary distributed nodes [24]. Shape functions can be created using RBFs and nodes in local support domains for function interpolation at any point in the local support domain.



Figure 1. Local support domains used in meshfree methods.

Consider a field function $u(\mathbf{x})$ defined in a problem domain Ω . A local support domain of an interest point \mathbf{x}_Q determines the vicinity nodes that are used for approximation or interpolation of function value at \mathbf{x}_Q . A support domain can have different shapes and its dimension and shape can be different from point to point, as shown in Figure 1. Most often used shapes are circular or rectangular. In this study, the number of field nodes (both regular and irregular) in the local support domain is predefined, i.e. n. According to the different distances between the field nodes and the point of interest \mathbf{x}_Q , the n nodes which are the nearest to the point of interest are adopted in the support domain. Then, the value of the field function $u(\mathbf{x})$ at interest point \mathbf{x}_Q can be approximated by interpolating the values of the field function at the vicinity nodes in the local support domain (see, e.g. References [1, 2])

$$u^{h}(\mathbf{x}, \mathbf{x}_{Q}) = \sum_{i=1}^{n} R_{i}(\mathbf{x})a_{i}(\mathbf{x}_{Q}) + \sum_{j=1}^{m} p_{j}(\mathbf{x})b_{j}(\mathbf{x}_{Q}) = \mathbf{R}^{\mathrm{T}}(\mathbf{x})\mathbf{a}(\mathbf{x}_{Q}) + \mathbf{p}^{\mathrm{T}}(\mathbf{x})\mathbf{b}(\mathbf{x}_{Q})$$
(1)

where *n* is number of nodes in the support domain, *m* is the number of terms of monomial (polynomial basis). $\mathbf{a}(\mathbf{x}_Q)$ is the vector of coefficients of radial basis function $\mathbf{R}^{\mathrm{T}}(\mathbf{x})$, and $\mathbf{b}(\mathbf{x}_Q)$ is the vector of coefficients of polynomial basis function $\mathbf{p}^{\mathrm{T}}(\mathbf{x})$, which are in the form of,

$$\mathbf{a}^{\mathrm{T}}(\mathbf{x}_{Q}) = \{a_1 \quad a_2 \quad a_3 \quad \cdots \quad a_n\}$$
(2)

$$\mathbf{b}^{\mathrm{T}}(\mathbf{x}_{O}) = \{b_{1} \quad b_{2} \quad b_{3} \quad \cdots \quad b_{m}\}$$
(3)

The polynomial terms here have to satisfy extra constraints to guarantee the unique approximation as [27],

$$\sum_{i=1}^{n} p_j(\mathbf{x}_i) a_i = \mathbf{P}_m^{\mathrm{T}} \mathbf{a} = 0, \quad j = 1, 2, \dots, m$$
(4)

There are number of radial basis functions available. Their properties and characteristics with arbitrary real shape parameters are well studied [1, 2]. Typical conventional radial basis functions are listed in Table I, where r_i is the distance between the interpolation point at **x** and a neighbourhood node at **x**_i.

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Туре	Expression	Dimensionless parameters	
Multi-quadrics (MQ)	$R_i(x, y) = (r_i^2 + (\alpha_c d_c)^2)^q$	$\alpha_c \ge 0, q$	
Gaussian (EXP)	$R_i(x, y) = \exp(-cr_i^2)$	С	
Thin plate spline (TPS)	$R_i(x, y) = r_i^{\eta}$	η	
Logarithmic RBF	$R_i(x, y) = r_i^{\eta} \log r_i$	η	

Table I. Typical radial basis functions with dimensionless shape parameters.

In this paper, the MQ radial basis function augmented with polynomials of completed first order is used in the radial point interpolation as,

$$\mathbf{p}^{\mathrm{T}}(\mathbf{x}) = \begin{bmatrix} 1 & x & y \end{bmatrix}$$
(5)

Enforcing the interpolation passing through the value at all nodes in the supporting domain leads to the following equation:

$$\mathbf{U}_{S} = \mathbf{R}_{Q}\mathbf{a}(\mathbf{x}_{Q}) + \mathbf{P}_{m}\mathbf{b}(\mathbf{x}_{Q}) \tag{6}$$

with constraint of

$$\mathbf{P}_m^{\mathrm{T}} \mathbf{a}(\mathbf{x}_Q) = 0 \tag{7}$$

where the vector of function values is

$$\mathbf{U}_S = \{u_1 \quad u_2 \quad \cdots \quad u_n\}^{\mathrm{T}} \tag{8}$$

the moment matrix of radial basis functions is

$$\mathbf{R}_{Q} = \begin{bmatrix} R_{1}(r_{1}) & R_{2}(r_{1}) & \cdots & R_{n}(r_{1}) \\ R_{1}(r_{2}) & R_{2}(r_{2}) & \cdots & R_{n}(r_{2}) \\ \vdots & \vdots & \ddots & \vdots \\ R_{1}(r_{n}) & R_{2}(r_{n}) & \cdots & R_{n}(r_{n}) \end{bmatrix}_{(n \times n)}$$
(9)

The polynomial moment matrix is

$$\mathbf{P}_{m} = \begin{bmatrix} 1 & x_{1} & y_{1} & \cdots & p_{m}(\mathbf{x}_{1}) \\ 1 & x_{2} & y_{2} & \cdots & p_{m}(\mathbf{x}_{2}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n} & y_{n} & \cdots & p_{m}(\mathbf{x}_{n}) \end{bmatrix}_{(n \times m)}$$
(10)

In Equation (9), r_k in $R_i(r_k)$ is defined as

$$r_k = \sqrt{(x_k - x_i)^2 + (y_k - y_i)^2}$$
(11)

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Combining Equations (6) and (7), we obtain the following expression:

$$\widetilde{\mathbf{U}}_{S} = \begin{bmatrix} \mathbf{U}_{S} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{Q} & \mathbf{P}_{m} \\ \mathbf{P}_{m}^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{a}(\mathbf{x}_{Q}) \\ \mathbf{b}(\mathbf{x}_{Q}) \end{bmatrix} = \mathbf{G}\mathbf{a}_{0}$$
(12)

where

$$\mathbf{a}_0^{\mathrm{T}} = \{a_1 \quad a_2 \quad \cdots \quad a_n \quad b_1 \quad b_2 \quad \cdots \quad b_m\}$$
(13)

$$\widetilde{\mathbf{U}}_S = \{u_1 \quad u_2 \quad \cdots \quad u_n \quad 0 \quad 0 \quad \cdots \quad 0\}$$
(14)

$$\mathbf{G} = \begin{bmatrix} \mathbf{R}_{\mathcal{Q}} & \mathbf{P}_m \\ \mathbf{P}_m^{\mathrm{T}} & \mathbf{0} \end{bmatrix}$$
(15)

Because the matrix \mathbf{R}_Q is symmetric, the matrix \mathbf{G} thus is symmetric as well. By solving Equation (12), one can have

$$\mathbf{a}_0 = \begin{cases} \mathbf{a} \\ \mathbf{b} \end{cases} = \mathbf{G}^{-1} \widetilde{\mathbf{U}}_S \tag{16}$$

Substituting Equation (16) into Equation (1) leads to

$$u^{h}(\mathbf{x}) = \{ \mathbf{R}^{\mathrm{T}}(\mathbf{x}) \quad \mathbf{P}^{\mathrm{T}}(\mathbf{x}) \} \mathbf{G}^{-1} \widetilde{\mathbf{U}}_{\mathcal{S}} = \widetilde{\mathbf{\Phi}}^{\mathrm{T}}(\mathbf{x}) \widetilde{\mathbf{U}}_{\mathcal{S}}$$
(17)

where

$$\widetilde{\boldsymbol{\Phi}}^{\mathrm{T}}(\mathbf{x}) = \{ \mathbf{R}^{\mathrm{T}}(\mathbf{x}) \quad \mathbf{P}^{\mathrm{T}}(\mathbf{x}) \} \mathbf{G}^{-1}$$
$$= \{ \phi_1(\mathbf{x}) \quad \phi_2(\mathbf{x}) \quad \cdots \quad \phi_n(\mathbf{x}) \quad \phi_{n+1}(\mathbf{x}) \quad \cdots \quad \phi_{n+m}(\mathbf{x}) \}$$
(18)

Finally, the RPIM shape functions corresponding to the nodal displacements vector $\Phi(\boldsymbol{x})$ are obtained as

$$\boldsymbol{\Phi}^{\mathrm{T}}(\mathbf{x}) = \{ \phi_1(\mathbf{x}) \quad \phi_2(\mathbf{x}) \quad \cdots \quad \phi_n(\mathbf{x}) \}$$
(19)

Equation (17) can be rewritten as

$$u^{h}(\mathbf{x}) = \mathbf{\Phi}^{\mathrm{T}}(\mathbf{x})\mathbf{U}_{S} = \sum_{i=1}^{n} \phi_{i}u_{i}$$
(20)

The derivative of the unknown variable function can be evaluated easily by differentiating Equation (20) as

$$u_{,l}^{h}(\mathbf{x}) = \mathbf{\Phi}_{,l}^{\mathrm{T}}(\mathbf{x})\mathbf{U}_{s}$$
(21)

where l denotes the coordinates either x or y. A comma designates a partial differentiation with respect to the indicated spatial coordinate that follows.

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RFDM FOR MECHANICS PROBLEMS

2.2. Radial point collocation method

Consider a partial differential governing equation defined in a domain Ω bounded by $\Gamma = \Gamma_t + \Gamma_u$:

$$A(\mathbf{u}) = 0 \tag{22}$$

where A() is a differential operator. One can discretize Equation (22) by simply collocating the differential equations at each node in the internal domain, using Equation (20) where the shape functions are created using local nodes, namely

$$A(\mathbf{u}_i) = 0 \tag{23}$$

The conditions on the Neumann boundary Γ_t is given as

$$B(\mathbf{u}) = 0 \tag{24}$$

where B() is the differential operator which defines the Neumann boundary conditions on Γ_t . Using Equation (20), Equation (24) can also be discretized at the nodes along Neumann boundary Γ_t as

$$B(\mathbf{u}_i) = 0 \tag{25}$$

With Equations (23) and (25), a set of algebraic equations can be obtained and expressed in the matrix form

$$\mathbf{KU} = \mathbf{F} \tag{26}$$

where **K** denotes the stiffness matrix, **U** is the vector of unknown variables at all nodes in the problem domain and \mathbf{F} is the nodal force vector.

Dirichlet boundary conditions are directly imposed at the nodes on Dirichlet boundary Γ_u in last stage as

$$\mathbf{u}_i = \overline{\mathbf{u}}_i \tag{27}$$

The final expression of the discretized system equations can be expressed in the following matrix form:

$$\overline{\mathbf{K}}\mathbf{U} = \overline{\mathbf{F}} \tag{28}$$

where $\overline{\mathbf{K}}$ is the final stiffness matrix, U is a vector of unknown variables at all nodes in the problem domain and $\overline{\mathbf{F}}$ is final nodal force vector.

The values of unknown variables at all nodes can be obtained by solving the resulting set of algebraic equations (28), if $\overline{\mathbf{K}}$ is not singular and well-conditioned. It is often found that $\overline{\mathbf{K}}$ behave far from well, partially because the RPCM uses local nodes for interpolation, which leads to possible instability in the solutions. It should be noted that the final stiffness matrix $\overline{\mathbf{K}}$ is generally not symmetric in the collocation methods.

3. FORMULATION OF THE RFDM

In this section, the RFDM is formulated. A general frame for constructing difference schemes is first proposed. Then a least-square technique is adopted to solve the differential equations.



Figure 2. Background grids for finite difference used in the RFDM.

Consider now a problem with a field variable U governed by a second-order partial differential equation as

$$A\frac{\partial^2 U}{\partial x^2} + B\frac{\partial^2 U}{\partial x \partial y} + C\frac{\partial^2 U}{\partial y^2} + D\frac{\partial U}{\partial x} + E\frac{\partial U}{\partial y} + FU + G = 0 \quad \text{in } \Omega$$
(29)

with boundary conditions

$$a\frac{\partial U}{\partial x} + bU = f \quad \text{on } \Gamma \tag{30}$$

where A, B, C, D, E, F, G, a, b and f are given constants or functions of x and y, Ω is the problem domain, Γ is the boundary of domain Ω . As shown in Figure 2, the dashed lines are background grids for regular rectangular mesh. The circles are field nodes and the black dots are finite difference (FD) grid points. There are M field nodes that carry the field variable U, and N FD grid points in the problem domain. We require N > M.

The classical central finite difference formulas are as follows (e.g. Reference [28]):

$$\left. \frac{\partial U}{\partial x} \right|_{i,j} = \frac{1}{2h} (U_{i+1,j} - U_{i-1,j}) \tag{31}$$

$$\left. \frac{\partial U}{\partial y} \right|_{i,j} = \frac{1}{2k} (U_{i,j+1} - U_{i,j-1}) \tag{32}$$

$$\frac{\partial^2 U}{\partial x^2}\Big|_{i,j} = \frac{1}{h^2} (U_{i-1,j} - 2U_{i,j} + U_{i+1,j})$$
(33)

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$$\left. \frac{\partial^2 U}{\partial y^2} \right|_{i,j} = \frac{1}{k^2} (U_{i,j-1} - 2U_{i,j} + U_{i,j+1})$$
(34)

$$\left. \frac{\partial^2 U}{\partial x \partial y} \right|_{i,j} = \frac{1}{4hk} (U_{i-1,j-1} - U_{i-1,j+1} + U_{i+1,j+1} - U_{i+1,j-1})$$
(35)

Substituting Equations (31)–(35) into Equation (29), we have

$$c_{1}U_{i-1,j-1} + c_{2}U_{i-1,j} + c_{3}U_{i-1,j+1} + c_{4}U_{i,j-1} + c_{5}U_{i,j} + c_{6}U_{i,j+1} + c_{7}U_{i+1,j-1} + c_{8}U_{i+1,j} + c_{9}U_{i+1,j+1} = -G$$
(36)

where

$$c_{1} = c_{9} = -c_{3} = -c_{7} = \frac{B}{4hk}, \quad c_{2} = \frac{A}{h^{2}} - \frac{D}{2h}, \quad c_{4} = \frac{C}{k^{2}} - \frac{E}{2k}$$

$$c_{5} = -2\left(\frac{A}{h^{2}} + \frac{C}{k^{2}}\right) + F, \quad c_{6} = \frac{C}{k^{2}} + \frac{E}{2k}, \quad c_{8} = \frac{A}{h^{2}} + \frac{D}{2h}$$
(37)

Here, h is assumed to be equal to k.

When the grid points approach the boundary Γ , the difference schemes in Equations (31)–(35) will not work. In such cases, we adopt, accordingly, backward difference, forward difference or both methods. In addition, when geometry of the boundary is very complex, we use simply the RPCM for the nodes on and near the boundary.

As described in Section 2, the value of field function U(x, y) at grid point (x_i, y_j) can be approximated by interpolating the values of the field function at the vicinity filed nodes in the local support domain. From Equation (20), we then have

$$U(x_{i}, y_{j}) = \sum_{l=1}^{n} \phi_{l} U_{l}$$
(38)

where ϕ_l is the value of the RPIM shape function at the local supporting field node, and U_l is the value of field function at the field node in the supporting domain. Similarly, the values of U(x, y) at other eight grid points in Equation (36) are obtained. Thus, after incorporating the values of field function at the grid points into the differential equation (36), a set of discretized governing equations can be obtained.

Following the procedures given by Equations (24)–(26), Neumann boundary conditions (30) can also be discretized at the nodes on Neumann boundary Γ_t .

Finally, a set of N algebraic equations can be obtained and expressed in the matrix form

$$\mathbf{K}_{(N \times M)} \mathbf{U}_{(M \times 1)} = \mathbf{F}_{(N \times 1)} \tag{39}$$

where $\mathbf{K}_{(N \times M)}$ denotes the stiffness matrix, $\mathbf{U}_{(M \times 1)}$ is the vector of unknown variables at all nodes in the problem domain Ω and $\mathbf{F}_{(N \times 1)}$ is the nodal force vector.

To solve the discretized equations (39), a least-square technique is utilized

$$\mathbf{K}_{(M \times N)}^{\mathrm{T}} \mathbf{K}_{(N \times M)} \mathbf{U}_{(M \times 1)} = \mathbf{K}_{(M \times N)}^{\mathrm{T}} \mathbf{F}_{(N \times 1)}$$
(40)

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or

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$$\overline{\mathbf{K}}_{(M \times M)} \mathbf{U}_{(M \times 1)} = \overline{\mathbf{F}}_{(M \times 1)} \tag{41}$$

where $\overline{\mathbf{K}}_{(M \times M)} = \mathbf{K}_{(M \times N)}^{\mathrm{T}} \mathbf{K}_{(N \times M)}$ is the modified system matrix, and $\overline{\mathbf{F}}_{(M \times 1)} = \mathbf{K}_{(M \times N)}^{\mathrm{T}} \mathbf{F}_{(N \times 1)}$ is the modified nodal 'force' vector.

Dirichlet boundary conditions (30) (when a=0) are directly imposed at the nodes on the Dirichlet boundary Γ_u in the final stage.

Thus, the final expression of the discretized system equations can be written as follows:

$$\hat{\mathbf{K}}_{(M \times M)} \mathbf{U}_{(M \times 1)} = \hat{\mathbf{F}}_{(M \times 1)}$$
(42)

where $\hat{\mathbf{K}}_{(M \times M)}$ is the final system matrix, and $\hat{\mathbf{F}}_{(M \times 1)}$ is the final nodal 'force' vector.

It should be noted that the Neumann boundary conditions are imposed in the process of forming the stiffness matrix $\mathbf{K}_{(N \times M)}$ and nodal force vector $\mathbf{F}_{(N \times 1)}$ in Equation (39) before implementing the least-square procedure. Dirichlet boundary conditions are imposed in the final stage only after the modified system matrix $\overline{\mathbf{K}}_{(M \times M)}$ and modified nodal 'force' vector $\overline{\mathbf{F}}_{(M \times 1)}$ are formed through the least-square procedure.

It is clear that the system matrix $\overline{\mathbf{K}}_{(M \times M)}$ is symmetric and positive definite through the least-square procedure. Stable results can be then obtained using a standard linear equation solver, such as the Cholesky solver.

4. NUMERICAL EXAMPLES

To examine the proposed RFDM, intensive numerical studies are carried out. The RFDM is first applied to a Poisson's equation problem that has exact solution for validity. Then an internal pressurized hollow cylinder is further investigated. As the third example, an infinite plate with a circular hole subjected to a unidirectional tensile load is considered. A bridge pier subjected to a uniformly distributed pressure on the top is studied in the example 4. To demonstrate the robustness of the proposed method to all the problem domains with irregular shapes, a relatively complicated triangle dam is tested as the last example. In this work, the MQ radial basis function augmented with linear polynomial function is used in computing RPIM shape functions. The dimensionless parameters (see, Table I) for the MQ radial basis function are taken as $\alpha_c = 4.0$ and q = 1.03. In the numerical studies, an error indicator is defined as follows:

$$e = \sqrt{\frac{\sum (u^{\text{numerical}} - u^{\text{exact}})^2}{\sum (u^{\text{exact}})^2}}$$
(43)

4.1. Poisson's equation

The proposed RFDM is first examined through solving a two-dimensional Poisson's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \sin(\pi x)\sin(\pi y)$$
(44)

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Figure 3. 100 regular field nodes (\bullet) and 441 finite difference grid points (\times).

Points	RPCM	FDM	RFDM	Exact
A(0.5, 0.5) B(0.95, 0.5) C(0.5, 0.3)	-0.05088411 -0.00786775 -0.04112709	-0.05076489 -0.00794138 -0.04106966	-0.05071812 -0.00791603 -0.04099582	$\begin{array}{r} -0.05066059 \\ -0.00792506 \\ -0.04098528 \end{array}$

Table II. Computed results of Poisson's equation.

The problem domain is $\Omega = \{(x, y) \in [0, 1; 0, 1]\}$. The exact solution is

$$u(x, y) = -\frac{1}{2\pi^2} \sin(\pi x) \sin(\pi y)$$
(45)

Dirichlet boundary conditions and Neumann boundary conditions are considered in the following studies.

4.1.1. Comparison study. To validate the present RFDM, we start with regular distribution of 10×10 field nodes, as shown in Figure 3. The background grid for finite difference is 21×21 regular rectangular mesh. A Dirichlet boundary is considered here, that is, the essential boundary conditions are imposed on all edges as

$$u = 0$$
 along $x = 0, 1, y = 0, 1$ (46)

Both RFDM and RPCM use 20 vicinity nodes in the local support domain for interpolation. Classical FDM is also used to solve this problem with the regular background grid. The comparisons at some selected nodes are listed in Table II. It can be found that the results obtained by the present RFDM are more accurate than those by both RPCM and FDM.

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Figure 4. Distribution of 121 extremely irregular nodes for Poisson's equation.



Figure 5. Result along the line of x = 0.5 for Poisson's equation.

To compare the present RFDM with RPCM, a critically irregular distribution of field nodes is employed and shown in Figure 4, where there are 121 irregular nodes in the domain. More than 30 nodes concentrate in one corner of the domain. The background grid is still 21×21 regular mesh. Twenty vicinity field nodes are used as the local supporting nodes. The results along x = 0.5 and y = 0.5 are plotted, respectively, in Figures 5 and 6. Results obtained by the RPCM are relatively inaccurate. However, the RFDM is able to provide results quite close to



Figure 6. Result along the line of y = 0.5 for Poisson's equation.

the exact solutions using even such an extremely irregular node distribution. The use of finite difference grids has clearly and significantly improved the stability of the RPCM.

In the RFDM, finite difference schemes based on regular grids are used to discretize the governing equations, which have been proven stable in general. The RPIM shape functions created using nodes in local support domains can be a source of instability. However, the RPIM shape functions are used for function interpolation only, no derivatives of RPIM shape functions are used, and hence magnification of error by differentiation is avoided (e.g. Reference [22]). Therefore, the stability is significantly improved.

4.1.2. Numerical convergence test. The convergence studies are conducted using the same boundary conditions as Equation (46). In Figure 7, four distributions of irregular nodes are shown. They are 50, 100, 200 and 400 field nodes, respectively. Twenty vicinity nodes are used for creating shape functions. The overall error norm of field variable u is obviously improved from 0.11% to 0.0095%, as shown in Figure 8 and Table III.

4.1.3. Parameter study. There are two important parameters used in the present RFDM: the numbers of local supporting nodes and finite difference grid points. The first one has been well investigated [1, 2, 19-21]. In this paper, the relations between the numbers of finite difference grid points and field nodes are discussed in details.

In this study, 20 vicinity nodes are fixed as the local supporting nodes for interpolation. The following mixed boundary conditions are considered in problem domain Ω , where Neumann boundary conditions are

$$\left. \frac{\partial u}{\partial x} \right|_{x=0} = -\frac{1}{2\pi} \sin(\pi y), \quad \left. \frac{\partial u}{\partial x} \right|_{x=1} = \frac{1}{2\pi} \sin(\pi y) \tag{47}$$

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Figure 7. Node distributions for Poisson's equation: (a) 50; (b) 100; (c) 200; and (d) 400 nodes.



Figure 8. Error norms of solution for Poisson's equation.

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Table III. Error norms of solution for Poisson's equation.



Figure 9. Optimal grid points for Poisson's equation.

Table IV. Optimal grid points for Poisson's equation.

No. of field nodes (M)	225	441	900
No. of optimal grid points (N)	576	1024	2209
Ratio (N/M)	2.56	2.32	2.45

and Dirichlet boundary conditions are

$$u = 0 \quad \text{along } y = 0, 1 \tag{48}$$

Three distributions of regular field nodes are used in the investigation: 15×15 , 21×21 and 30×30 . The 'optimal' numbers of grid points (N) with respect to the corresponding field nodes (M) for Poisson's equation are shown in Figure 9 and Table IV. It is found that the ratio of N/M should be between 2 and 3. In Equation (39), there are M unknown variables and N algebraic equations. To get the solutions, N should not be less than M. On the other hand, if N becomes too large, the M unknown variables will be over smoothed through the least-square procedure. Namely, the solutions will be inaccurate.

4.2. Internal pressurized hollow cylinder

A hollow cylinder under internal pressure shown in Figure 10 is now used in the benchmark study. The parameters are taken as internal pressure p = 100 Pa, shear modulus G = 8000 Pa

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Figure 10. Hollow cylinder subjected to internal pressure.

and Poisson's ratio v = 0.25. This problem was studied by several other researchers [29, 30] as a benchmark problem, since the analytical solution is available. The exact solutions of radial and circumferential stresses are

$$\sigma_r = \frac{a^2 p}{b^2 - a^2} \left(1 - \frac{b^2}{r^2} \right) \tag{49}$$

$$\sigma_{\theta} = \frac{a^2 p}{b^2 - a^2} \left(1 + \frac{b^2}{r^2} \right) \tag{50}$$

The radial displacement for plane strain problem is

$$u_r = \frac{(1-v^2)r}{2(1+v)G} \left(\sigma_\theta - \frac{v}{1-v}\sigma_r\right)$$
(51)

where r is the radial coordinate, a is the inner radius and b is the outer radius.

4.2.1. Comparison study. Due to the symmetry of the problem, only one-quarter of the cylinder needs to be modelled. As shown in Figure 11, there are 95 nodes irregularly distributed in this problem domain. In the RFDM, 24 vicinity nodes are used in the support domain.

The RFDM results are compared with the FEM results and analytical solutions. In this study, commercial FEM software, ANSYS, is used to compute the FEM results. In the FEM model, the same set of nodes in Figure 11 is used. Triangle element is adopted in the FEM computation. The radial displacement, circumferential stress and radial stress along the line of y = x are plotted in Figures 12–14, respectively. It can be found that the RFDM results are in very good agreement with the exact solutions. In comparison with FEM results, the radial



Figure 11. Node distribution for the hollow cylinder.



Figure 12. Radial displacement u_r along the line of y = x in the hollow cylinder.

displacement and circumferential stress by RFDM are generally more accurate than those of FEM.

4.2.2. Numerical convergence test. The convergence studies are conducted using three different nodal densities (200, 400 and 800 irregular nodes), as shown in Figure 15. Twenty-one vicinity nodes are used for interpolation. The overall error norm of radial displacement u_r has been

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Figure 13. Circumferential stress σ_{θ} along the line of y = x in the hollow cylinder.



Figure 14. Radial stress σ_r along the line of y = x in the hollow cylinder.

improved a lot from 1.66% to 0.45%, as shown in Figure 16 and Table V. A very linear steady convergence is observed.

4.2.3. Parameter study. In this study, 21 vicinity nodes are fixed as the local supporting nodes for interpolation. As shown in Figure 15, three distributions of irregular field nodes are investigated. The 'optimal' numbers of grid points (N) with respect to the corresponding field



Figure 15. Node distributions in the hollow cylinder: (a) 200; (b) 400; and (c) 800 nodes.



Figure 16. Error norms of solution for hollow cylinder.

Table V. Error norms of solution for internal pressurized hollow cylinder.

No. of field nodes	200	400	800
Error norm	1.6565E-002	8.5961E-003	4.4910E-003

nodes (M) for hollow cylinder are shown in Figure 17 and Table VI. It is also found that the ratio of N/M is between 2 and 3.

According to the above parameter investigations made for both Poisson's equation and internal pressurized hollow cylinder, the relationship between M and N is proposed as follows:

$$N \in [2M, 3M] \tag{52}$$

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Figure 17. Optimal grid points for internal pressurized hollow cylinder.

Table VI. Optimal grid points for internal pressurized hollow cylinder.

No. of field nodes (M)	200	400	800
No. of optimal grid points (N)	498	970	1881
Ratio (N/M)	2.49	2.43	2.35

4.3. Infinite plate with a circular hole

To validate the RFDM in simulating stress concentration, we consider an infinite plate with a central circular hole subjected to a unidirectional tensile load of 1.0 in the x direction. Due to the symmetry, only the upper right quadrant of the plate is modelled, as shown in Figure 18. The plane strain problem is considered, and the geometries and material parameters used are a = 1, b = 5, Young's modulus $E = 1.0 \times 10^3$ and Poisson's ratio v = 0.3. Symmetry conditions are imposed on the left and bottom edges, and the inner boundary of the hole is traction free. The exact solutions for the stresses in the plate are given in the polar coordinate [31]:

$$\sigma_{xx} = 1 - \frac{a^2}{r^2} \left(\frac{3}{2} \cos 2\theta + \cos 4\theta \right) + \frac{3}{2} \frac{a^4}{r^4} \cos 4\theta$$

$$\sigma_{xy} = -\frac{a^2}{r^2} \left(\frac{1}{2} \sin 2\theta + \sin 4\theta \right) + \frac{3}{2} \frac{a^4}{r^4} \sin 4\theta$$

$$\sigma_{yy} = -\frac{a^2}{r^2} \left(\frac{1}{2} \cos 2\theta - \cos 4\theta \right) - \frac{3}{2} \frac{a^4}{r^4} \cos 4\theta$$
(53)

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Figure 18. Quarter model of the infinite plate with a circular hole subjected to a unidirectional tensile load.

where (r, θ) are the polar coordinates and θ is measured counterclockwise from the positive x-axis. Traction boundary conditions given by the exact solution (53) are imposed on the right (x=5) and top (y=5) edges.

Figure 19 shows the node distribution in the problem domain, in which there are 366 field nodes and 987 background grid points. Twenty-three vicinity nodes are used in the support domain. The distribution of stress σ_{xx} at x = 0 obtained using the RFDM is shown in Figure 20. It can be observed from this figure that the RFDM yields satisfactory results for the problem.

4.4. Bridge pier

In this example, the RFDM is used for the stress analysis of a bridge pier subjected to a uniformly distributed pressure on the top, as shown in Figure 21. The problem is solved as a plain strain case with material properties $E = 4 \times 10^{10}$ Pa, v = 0.15 and loading $P = 10^5$ Pa.

Due to the symmetry, only right half of the bridge is modelled as shown in Figure 22 where there are 386 field nodes and 995 background grid points in the model and 30 vicinity nodes are used in the support domain. In this study, the FEM results are computed with ANSYS using quadratic element and the same set of nodes in Figure 22. Comparison of the stress distribution σ_{yy} computed by the RFDM and the FEM are shown in Figure 23. The results computed by the RFDM are in good agreement with the results computed by the FEM.

4.5. Triangle dam of complicated shape

As the last example, to generalize the present RFDM to all problem domains with irregular shapes, a triangle dam with complicated geometry subjected to a uniformly distributed pressure



Figure 19. Node distribution of the infinite plate with a circular hole: 366 field nodes (dots) and 987 grid points (intersections of dashed lines).



Figure 20. Normal stress σ_{xx} along the edge of x = 0 in a plate with a central hole subjected to a unidirectional tensile load.

on the surface is studied, as shown in Figure 24(a). The problem is treated as the plane strain case with the same material properties as in the bridge pier mentioned above. Due to symmetry, only the right half of the dam is simulated. The geometry of the triangle dam is shown in Figure 24(b).



Figure 21. A bridge subjected to a uniformly distributed pressure on the top.



Figure 22. Nodal distribution in the bridge model: 386 field nodes (dots) and 995 grid points (intersections of dashed lines).

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Figure 23. Distribution of normal stress σ_{yy} in the bridge: (a) RFDM; and (b) ANSYS.



Figure 24. A triangle dam subjected to uniformly distributed pressure on the surface.

Figure 25(a) shows the node distribution of 334 irregular field nodes (dots), where there are 742 background grid points (intersections of dashed lines) in the problem domain and 24 vicinity nodes in the local supporting domain for the present RFDM. For comparison, commercial FEM software, ANSYS, is used to compute the FEM results with quadratic element and same set of nodes in Figure 25(a). Since no analytical solution is available for this problem, a reference solution is obtained by ANSYS using a very fine mesh of 4462 irregular nodes, as shown in Figure 25(b).



Figure 25. Node distributions in the triangle dam: (a) 334; and (b) 4462 field nodes.



Figure 26. Displacements along the line of x = 8: (a) x direction; and (b) y direction.

Figure 26 shows the displacement components along the line of x = 8. It can be found that the RFDM results are more accurate than those of FEM, according to the FEM reference solutions. The stress distribution σ_{yy} by RFDM is plotted in Figure 27(a). The result obtained using ANSYS with the same nodes as RFDM is shown in Figure 27(b). Figure 27(c) is the reference result for σ_{yy} . It can be concluded that the RFDM results are accurate enough for general engineering requirement.



Figure 27. Normal stress σ_{yy} distribution: (a) RFDM; (b) ANSYS; and (c) ANSYS (Reference).

5. CONCLUSIONS

In this paper, a radial point interpolation based finite difference method (RFDM) has been presented for solving partial differential equations, with the emphases on solid mechanics problems. By incorporating the radial point interpolation into the classical finite difference approach, the proposed RFDM overcomes the instability of radial point collocation method. The use of a least-square technique helps further to obtain a system matrix with good properties and the resultant set of algebraic equations can be solved more efficiently and accurately by standard solver such as Cholesky solver. A large number of numerical examples are studied and some important parameters are investigated in detail.

From the research work conducted, the following conclusions can be drawn:

1. Shape function generated using RBFs augmented with polynomials possess the Delta property, which allows the essential boundary conditions to be enforced directly.

- 2. From the comparison studies with the radial point collocation method, it is found that the RFDM has good stability due to the use of finite difference grids to get the discrete system equations. Shape functions constructed using local supporting nodes are used for function interpolation only (not the derivatives).
- 3. Based on the study of examples in this paper, the relationship between field nodes M and the corresponding finite difference grid points N is recommended as N = (2-3) M for the RFDM.
- 4. From the stress analyses of several numerical examples, the RFDM provides very good results compared to even the well-established finite element method.
- 5. As demonstrated in this work, the RFDM can be applied with good performance to problem domains of irregular shapes.

In summary, we conclude that the RFDM is a stable, robust and reliable numerical method based on strong-form formulation for mechanics problems.

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